Philosophy 324A
Philosophy of Logic
2016

Note Six

A COUPLE OF WRAP-UP REMARKS ABOUT THE CONCEPT CLARIFICATION SPECTRUM

1. *Sets*: Consider the ordinary notion of set, as in a set of china or the smart set. Everyone understands that sets are made up by their members. With just one exception, Frege's original axioms of 1884 and, later 1893, serve as the *analysis* of the ordinary concept and also as an *explication.* The empty set axiom would be the outlier, by amending the concept of set to allow for a bona fide set defined by the condition that nothing whatever is in it. The empty set axiom *rationally reconstructed* the intuitive concept.

   In 1902, Russell communicated the following proof to Frege, establishing what came to be called

RUSSELL'S PARADOX

1. Consider the predicate "is a nonselfmembered set.
2. By the comprehension axiom (Frege's Basic Law V), there exists a set whose members are all and only those sets that aren't members of themselves. Call this set R.
3. Either R is a member of itself or it is not (by the law of excluded middle).
4. If R is a member of itself, it is not nonselfmembered, hence not a member of itself.
5. On the other hand, if R isn't a member of itself, it meets a condition necessary and sufficient for membership, hence is a member of itself.
6. Therefore, there exists a set which both is and isn't a member of itself.

The first post-paradox treatment of sets was Russell's own *Principles of Mathematics* in 1903. In that work, he insisted that the intuitive concept of set was philosophically unanalyzable, and that since it certainly wasn't a primitive concept he concluded that there really wasn't *any* intuitive concept at all. In other words, the predicate of mathematical English "is a set" has a *null* extension. Nothing whatever is a set in that sense. And yet, modern arithmetic is undoable without a workable concept of set. So *something* would have to be conjured up to make arithmetic possible. And Russell emphasized that this new concept would have to be introduced by *nominal definition*, that is, would have to be made up. That, of course, would be an example of *stipulation.*

2. *Logicism*

Logicism is a philosophical doctrine about arithmetic. It is designed to assuage long-standing philosophical worries about mathematics in general – e.g. whether mathematical objects really exist and, if they are, how is our knowledge of them possible, given that we stand in no *palpable* relations to them? Logicism is also an effort to calm the waters of the transfinite. There are two different sources of worry. One is whether anything in reality actually instantiates the Dedikind definition of an infinitely-membered set as a set that bears a one-to-one correspondence to at least one of its own proper subsets. The other bone of contention isCantor's famous *diagonal proof* that some infinite sets are larger than other infinite sets – e.g. the number of real numbers is greater than the number of the natural numbers. The worry was whether any real objects actually satisfy the conditions of this proof.

Frege also had what he thought of as mathematical reservations about the intellectual integrity of mathematics itself, e.g. the imaginary numbers such as π which is equivalent to no real number nor any natural one either. By the time of his *habilitation* dissertation in (I think) 1878, Frege had satisfied himself that all of mathematics could be safely represented in arithmetic, that is, number theory. But he wasn't able to bring himself to believe that arithmetic was capable of furnishing its own intellectually secure foundations. Unless its foundations could be found somewhere else, the whole edifice of mathematics was at risk of collapse. *Note well*: all this is before the discovery of Russell's paradox of sets.

Frege and Russell had come independently to the view that each of the particular problems that bugged them separately would evaporate once a safe means of formally representing arithmetic was found. They both agreed that that a safe home would be *pure logic.* So now the trick was to find the logic that would turn this trick, namely what we now call CQT supplemented by the pre-paradox axioms for sets. Frege's *Begriffsschrift* of 1879 would set out the complex apparatus of formalization. The *Grundlagen* of 1884 (*Foundations of Arithmetic*) would give an informal exposition of how the reduction of arithmetic to pure logic is actually achieved. A more precise, rigorous and technically powerful treatment appeared in 1893 in volume I of the *Grundgesetze* (*The Basic Laws of Arithmetic*). Volume II would appear in 1903, and contained a doleful Appendix announcing the paradox that Russell has communicated to him just scant months before. This is the same year, of course, in which Russell's *Principles* also appears. It, too, contained an Appendix, indeed two of them. One was on Meinong's theory of objects. The other was on Frege's now failed foundational project for arithmetic. *Principles* also sketched what appears to have been the first post-paradox attempt to get logicism to work, by way of what Russell would call *the theory of types.* (It wasn't inconsistent, but had problems of its own.)

3. *Postscript*

Russell seems not to have been the discoverer of the paradox that bears his name. It may have been known to Cantor (1845-1918) and appears to have been known to Zermelo (1871-1953). Either way, no fuss was made. You could do perfectly useful set theory, paradox and all, or you could set out to find the *right* axioms for sets. This may strike us as perfectly sensible – the paradox is an embarrassment and a nuisance. But if it bothers us all that much, we can get to work and find some consistent axioms.

In fact, this marks a *hugely* important difference between how analytic philosophers of mathematics and mathematicians themselves responded to the annoyance of inconsistency. There is more about this in "Does changing the subject from A to B enlarge our understanding of A?" online from my webpage.

Finally, here is something on which I won't be examining you. Read it if you like; otherwise don't bother. After a good deal of to-ing and fro-ing the dominant post-paradox set theory not yet known to be inconsistent is ZF, evolving from axioms first laid down by Ernst Zermelo in 1908 and subsequently refined by Abraham Fraenkel (1891-1965), Thoralf

Skolem, Hermann Weyl (and others), now known as Zermelo-Fraenkel set theory. It has the axioms of extensionality, foundation, infinity, pairing, power set, replacement, separation and union. When the axiom of choice is added ZF becomes ZFC, Zermelo-Fraenkel set theory with choice. The point of it all is to represent sets as *iterative.* All sets obtain within a cumulative hierarchy, organized into levels by ordinal numbers, each level produced from preceding levels by the power set operation.

Russell would have said that ZF sets are what those axioms stipulate them to be. Zermelo *et al*. would have said, “no, the ZF axioms are, we think, the best approximation to date of what sets really are. Gödel (1906-1978) said the same.